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Comments on the quasi-normal Markovian approximation for fully-developed turbulence

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In a recent paper, Tatsumi, Kida & Mizushima (1978) have made a numerical study of the quasi-normal Markovian (QNM) equation for homogeneous isotropic incompressible turbulence at Reynolds numbers R up to 800.

Analytical investigations of the QNM equation support the contention of Tatsumi *et al.* that, at $R = \infty$, the decay of an initial energy spectrum of the form $k^a \exp(-k^2)$ leads to an initial energy-conserving regularity phase followed by a self-similar decay phase. During the former we give explicit expressions for the enstrophy and skewness. During the latter we show that for $1 < a < 4$ the energy follows, for $t \to \infty$, a t^{-b} law with the usual value $b = 2(a+1)/(a+3)$; when $a \geqslant 4$ deviations from Kolmogorov's (1941) $t^{-\frac{10}{7}}$ law originate from non-local 'beating' interactions between eddies with sizes of the order of the integral scale.

We also show, analytically, that the QNM equation has a k^{-2}, not a $k^{-\frac{5}{3}}$, inertial range and that its dissipation range is of the form $k^3 e^{-k/k_D}$, rather than $e^{-\sigma k^{1.5}}$.

Our results are illustrated by numerical integration of the QNM equation for R up to 10^6 and by comparison with results from the eddy-damped quasi-normal Markovian equation which is known to produce a $k^{-\frac{5}{3}}$ spectrum.

1. Introduction

The fundamental closed spectral equation used by Tatsumi *et al.* (1978; hereinafter referred to as TKM) may be obtained by the following procedure, which justifies our calling it the quasi-normal Markovian (QNM) approximation: (i) write the quasi-normal approximation assuming zero initial triple correlations, this leads to an equation for the rate of change of the energy spectrum $E(k,t)$ involving in the right-hand side a time integral over energy spectra at prior times s; (ii) Markovianize the time-integral by updating such spectra to time t. Using the notation of two recent review

† Work performed in part at the Division of Applied Sciences, Harvard University.

papers (Orszag 1976; Rose & Sulem 1978), TKM's equations (4.22) and (4.23) may then be written:

$$\frac{\partial E(k,t)}{\partial t} + 2\nu k^2 E(k,t) = T(k,t), \tag{1.1}$$

$$T(k,t) = \int_{\Delta_k} dp\, dq\, \theta_{kpq}(t) \frac{xy+z^3}{q} [k^2 E(p,t) E(q,t) - p^2 E(q,t) E(k,t)], \tag{1.2}$$

$$\theta_{kpq}(t) = [1 - \exp\{-\nu t(k^2+p^2+q^2)\}]/\nu(k^2+p^2+q^2). \tag{1.3}$$

The integral is over the strip Δ_k in the (p,q) plane such that k, p and q can form a triangle; x, y and z are the cosines of the interior angles, α, β, γ, of this triangle. Equivalence of (1.1), (1.2) and (1.3) with TKM's equations (4.22) and (4.23) is easily checked by making the changes of variables $E(k,t) = 4\pi k^2 \phi(k,t)$ and $(p,q) \rightarrow (p,z)$. (In TKM's notation p and q are k' and k'' and z is $-\mu$.)

Other approximations of the same class, with different choices of θ_{kpq}, have been discussed in the literature (Orszag 1970; Kraichnan 1971*a*,*b*; Sulem, Lesieur & Frisch 1975; André & Lesieur 1977; see also the above review papers and Leslie 1973). Usually the triad relaxation time θ_{kpq} contains, in addition to the viscous damping, an eddy damping calculated from a local eddy turnover time; such a choice referred to as eddy-damped quasi-normal Markovian (EDQNM) ensures compatibility with Kolmogorov's (1941) theory. It leads to a $k^{-\frac{5}{3}}$ inertial range. Furthermore, EDQNM (and not QNM) reproduces the correct exponents for one of the few turbulence problems which can be handled by systematic (renormalization-group) techniques, namely for the infra-red behaviour of a fluid with power law forcing (Forster, Nelson & Stephen 1977; Fournier & Frisch 1978; de Dominicis & Martin 1979; Sulem, Fournier & Pouquet 1979).

Compatibility with Kolmogorov's (1941) theory should not be considered as the crucial test of analytic theories of turbulence. Indeed, there are strong reasons to believe that, because of the intermittency in the small scales, high-Reynolds-number turbulence deviates from Kolmogorov's (1941) predictions. This possibility has been noticed by Kolmogorov (1962) himself and now plays a central role in theoretical work on high-Reynolds-number turbulence (Kraichnan 1974; Mandelbrot 1976; Rose & Sulem 1978; Frisch, Sulem & Nelkin 1978).

It must be stressed that both the QNM and the various EDQNM approximations are *realizable*: the energy spectrum satisfies the probabilistic positivity constraint $E(k,t) \geqslant 0$. This is proven, for example, in the appendix of Rose & Sulem (1978) for arbitrary positive θ_{kpq}.

The aim of the present paper is not to discuss the problem of what is the 'best' closure; we wish to concentrate on the *consequences* of the QNM equation.

TKM have studied the QNM equation numerically at Reynolds numbers up to 800. They claim among other results to have established the following.

(1) The high wavenumber domain consists of three parts (by increasing wavenumbers):

(*a*) a $k^{-\frac{5}{3}}$ inertial range;

(*b*) a k^{-1} range;

(*c*) an $\exp(-\sigma k^{1\cdot 5})$ dissipation range.

(2) The free decay of an initial spectrum of the form $k^a e^{-k^2}$ leads to an initial period, $0 \leqslant t \leqslant t_*$, during which the energy $E(t)$ is approximately conserved.

(3) For $t \gg t_*$ a power law is observed for $E(t)$:

(*a*) for $a = 2$ they obtain Saffman's (1967) $t^{-\frac{6}{5}}$ law.

(*b*) for $a = 4$ they find $E(t) \propto t^{-1.39}$ which is 'in agreement' with Kolmogorov's (1941) $t^{-\frac{10}{7}}$ law. They point out however that Kolmogorov's derivation seems doubtful because it is based on the constancy of the Loitsiansky integral.

We shall show that (i) results (1*a*) and (1*c*) do not survive at very high Reynolds numbers; (ii) in the limit of infinite Reynolds number the energy conservation result (2) becomes exact and t_* can be calculated analytically; (iii) for the case (3*b*) the departure from the $t^{-\frac{10}{7}}$ law is due to 'beating' interactions between eddies with sizes of the order of the integral scale contributing a k^4 term to the transfer integral at small k and thereby destroying the Loitsiansky invariant.

All these results will be obtained by analytical methods and checked by numerical integration at Reynolds numbers up to 10^6. Part of the material in the present paper is just an adaptation to the QNM equation of the study by Lesieur & Schertzer (1978; hereinafter referred to as LS) of the eddy-damped quasi-normal Markovian (EDQNM) equation, we shall therefore be rather brief and refer the reader to LS for some of the details.

2. The inertial solutions

At infinite Reynolds number ($\nu \downarrow 0$) the QNM triad relaxation time takes the simplified form:

$$\theta_{kpq} \equiv t \quad (\nu \downarrow 0). \tag{2.1}$$

The zero transfer (inertial) solutions of the QNM equation may then be obtained by the following procedure (Kraichnan 1971*b*; Fournier & Frisch 1978; Rose & Sulem 1978). Write the energy flux:

$$\Pi(k) = -\int_0^K T(k)\, dk. \tag{2.2}$$

Dimensional inspection of (1.2) (counting factors of t, of k and of E) then gives

$$[\Pi] = [t]\,[k]^4\,[E]^2. \tag{2.3}$$

Hence, constancy of the energy flux ($\Pi(k) \equiv \epsilon$) implies

$$E(k) \sim t^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}k^{-2}. \tag{2.4}$$

The dimensional analysis can be made rigorous by checking the convergence of the corresponding energy flux triple integral. Note that in the infinite-Reynolds-number limit the QNM differs only by a factor t from the Markovian random coupling model equation which is well known to produce a k^{-2} spectrum (Frisch, Lesieur & Brissaud 1974).†

It must be stressed that the existence of a finite rate of dissipation ϵ in the limit of infinite Reynolds numbers in no way implies a Kolmogorov spectrum, as soon as a

† The same analysis applied to two-dimensional turbulence leads, for the energy spectrum, to a k^{-3} enstrophy cascade and a k^{-2} inverse energy cascade (Lesieur 1973; see also Tatsumi & Yanase 1978).

dimensional quantity other than ϵ and k appears in the energy spectrum. This can be the integral scale in models of intermittent turbulence (Frisch *et al.* 1978), the r.m.s. velocity in Kraichnan's (1959) direct-interaction approximation, an externally introduced parameter τ_0 in the Markovian random coupling model (Frisch *et al.* 1974) or the time in the present QNM closure.

In the QNM at small but non-zero viscosity, and sufficiently large times, there is an addition to the k^{-2} inertial range a pseudo-inertial solution corresponding to the range (T for transition, D for dissipation)

$$K_T = (\nu t)^{-\frac{1}{2}} \ll k \ll k_D = (\epsilon/\nu^3)^{\frac{1}{4}}. \tag{2.5}$$

The first inequality ensures that $\nu k^2 t \gg 1$ so that the QNM triad relaxation time reduces to

$$\theta_{kpq} = 1/\nu(k^2+p^2+q^2). \tag{2.6}$$

The second inequality ensures that the dissipative term $2\nu k^2 E(k,t)$ may be neglected. Dimensional inspection (supported by checking convergence) gives

$$[\Pi] = [\nu]^{-1}[k]^2[E]^2 \tag{2.7}$$

and thus

$$E(k) \sim (\epsilon\nu)^{\frac{1}{2}} k^{-1}. \tag{2.8}$$

Note that, as observed by TKM, the quasi-normal approximation (not Markovianized) leads precisely to a k^{-2} inertial and k^{-1} pseudo-inertial solution. This was shown analytically by Tatsumi (1960); his derivation also applies, practically without change, to the QNM equation.

In order to check these analytic results we have integrated the QNM equation at high Reynolds numbers using a numerical method described in LS. Initial conditions are, within normalization factors, taken the same as case II of TKM, namely

$$E(k,0) \propto k^4 \exp\{-2(k/k_0)^2\}. \tag{2.9}$$

The Reynolds number is $R = v_0/\nu k_0 = 10^6$, where

$$\tfrac{1}{2}v_0^2 = \int_0^\infty E(k,0)\,dk \tag{2.10}$$

is the initial kinetic energy.

Minimum and maximum wavenumbers are $k_{\min} = 2^{-8}k_0$, $k_{\max} = 2^{15}k_0$. The number of points per wavenumber octave is $F = 4$. In figure 1 the energy spectrum is shown for $t = 0$ and $t = 5$. The k^{-2} and k^{-1} ranges are clearly displayed. In figure 2, we have represented similar results for $R = 800$, the highest value chosen by TKM. At such values of R we do not expect a clearcut k^{-2} range because the extension of the inertial range, something like a decade, is too small. More precisely, it is known that in wavenumber space the transfer integral has a 'range' of approximately one decade in each direction; i.e. most of the integral $T(k)$ comes from triads kpq with ratios max $(k,p,q)/\min(k,p,q)$ up to approximately 10 (Kraichnan 1971*b*; André & Lesieur 1977; Rose & Sulem 1978). Thus, at $R = 800$, the central wavenumber of the inertial range will be contaminated by the k^{-1} range which is less than one half decade away; therefore the inertial range will be somewhat shallower than k^{-2}.

Recent numerical results of Tatsumi & Kida (private communication) indicate that at very high Reynolds numbers the solution of the QNM equations displays a $k^{-\frac{5}{3}}$ law at wavenumbers just beyond the energy range and lower than the k^{-2} inertial range.

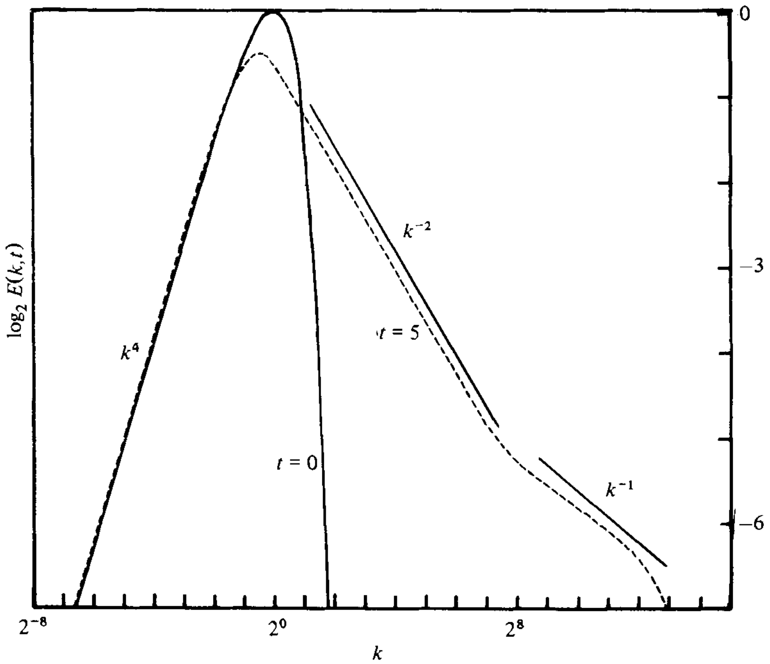

FIGURE 1. Evolution of the energy spectrum for the QNM approximation. The initial Reynolds number is $R = 10^6$. At $t = 5$ the spectrum displays a two-decade k^{-2} inertial range and a one-decade k^{-1} pseudo-inertial range.

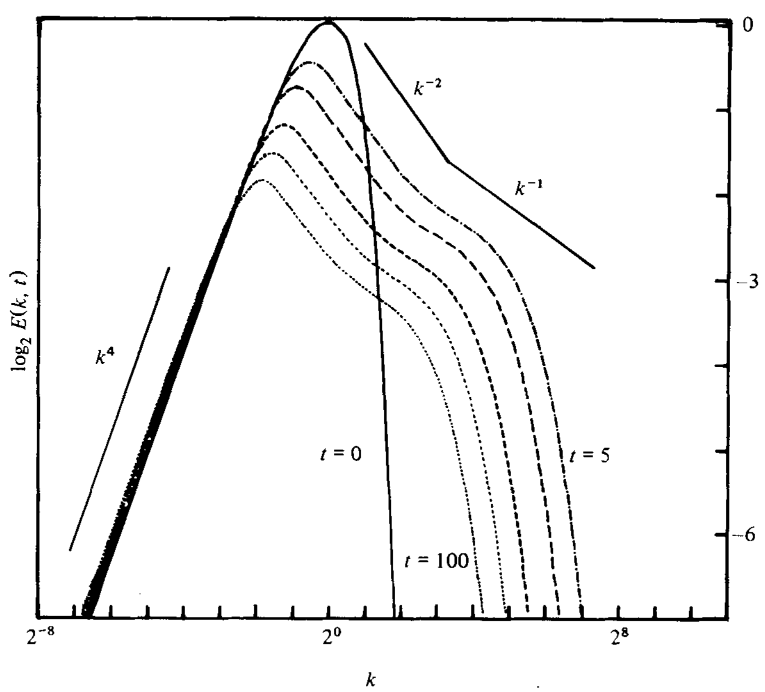

FIGURE 2. QNM approximation. The same conditions as in figure 1, except that here $R = 800$ as in TKM. The energy spectrum is shown at times $t = 0, 5, 10, 25, 50, 100$.

However, the extent of this 'range' does not seem to increase with the Reynolds numbers, contrary to the inertial ranges observed experimentally.

3. Decaying turbulence: The initial 'regularity' phase

TKM find numerically that there is 'an abrupt change in the behaviour of the energy spectrum' at a finite time. This is best seen on their figure 7 which shows a near constancy of the energy during a finite time.

This fact is easily demonstrated analytically. Indeed letting $\nu \downarrow 0$ on the QNM equation (1.1), θ_{kpq} reduces to t. Multiplying by k^2 and integrating over k from 0 to ∞ we obtain after tedious but very standard manipulations (Proudman & Redi 1954; Orszag 1976; Rose & Sulem 1978)

$$d\Omega/dt = \tfrac{2}{3}t\Omega^2, \tag{3.1}$$

where

$$\Omega(t) = \int_0^\infty k^2 E(k,t)\,dk \tag{3.2}$$

is the enstrophy. Hence, assuming an initially finite enstrophy,

$$\Omega(t) = \Omega(0)\,[1-\tfrac{1}{3}\Omega(0)\,t^2]^{-1}. \tag{3.3}$$

Enstrophy blows up at

$$t_* = [3/\Omega(0)]^{\frac{1}{2}}. \tag{3.4}$$

When a finite viscosity is included, it is easily checked that (3.1) holds with '$\leqslant$' instead of '$=$'. Hence, for $0 \leqslant t \leqslant T < t_*$, the enstrophy is bounded uniformly in t and ν; thus the energy dissipation $dE/dt = -2\nu\Omega$ tends to zero and energy is conserved.

We can also calculate the skewness during the initial phase in the limit $\nu \downarrow 0$. From TKM's equation (5.11) we have

$$S(t) = \frac{3\sqrt{30}}{14}\,\Omega^{-\frac{3}{2}}(t)\,d\Omega(t)/dt \tag{3.5}$$

$$= \frac{\sqrt{30}}{7}\,t\Omega^{\frac{1}{2}}(0)\,[1-\tfrac{1}{3}\Omega(0)\,t^2]^{-\frac{1}{2}}. \tag{3.6}$$

The skewness blows up like $(t_*-t)^{-\frac{1}{2}}$ as $t \uparrow t_*$. This explains the huge overshoot in the skewness observed by TKM (their figure 8). Notice that in EDQNM there is also an overshoot but a finite one as implied by the enstrophy inequality

$$d\Omega/dt \leqslant C\Omega^{\frac{3}{2}} \quad \text{(EDQNM)} \tag{3.7}$$

(see André & Lesieur's 1977 equation (2.11) and their figure 5).

4. Decaying turbulence: the self-similarity phase

For $T > t_*$ one can prove for the QNM that the enstrophy is, for $\nu > 0$ and $0 \leqslant t \leqslant T$ bounded uniformly in t but not in ν (immediate adaptation of the proof given in Rose & Sulem 1978, §6.2). TKM's and our calculations give strong numerical evidence that for $\nu \downarrow 0$ the dissipation tends to a finite limit and that there is a k^{-2} inertial range

$$E(t,k) \propto k^{-2}, \quad \nu \downarrow 0, \quad k \to \infty, \quad t > t_*. \tag{4.1}†$$

† When the QNM procedure is applied to Burgers' model this assertion can be proven for $t > t_{**} \geqslant t_*$. Indeed for $\nu \downarrow 0$ the QNM equation becomes the MRCM equation by taking t^2 as new time variable. One then uses theorem 3.6 of Bardos *et al.* (1979).

We now wish to concentrate on the behaviour for $t \gg t_*$. As explained by TKM the law of decay of the total energy depends crucially on the behaviour of the spectrum at small wavenumbers (very large eddies). An important feature of the large scale dynamics, not particularly stressed in TKM, is the non-localness of interactions: the transfer integral $T(k)$ for $k \ll k_0$ (k_0 = inverse of integral scale) is not determined by wavenumbers p, $q \sim k$ but comes mostly from very elongated triads

$$k \ll p \sim q \sim k_0. \tag{4.2}$$

Indeed, if a

$$E(k) \propto k^a \quad (a > 1) \tag{4.3}$$

law is used for the energy spectrum, the transfer integral is found to diverge at high wavenumbers (Fournier & Frisch 1978). When the k^a range does not extend to $k = \infty$ the dominant contribution may be obtained by expanding the transfer integral in powers of k/k_0. To leading order one obtains (cf. LS for details)

$$T(k,t) = \tfrac{14}{15} k^4 \int_0^\infty \theta_{kpp}(t) \frac{E^2(p,t)}{p^2} dp + O\left[\left(\frac{k}{k_0}\right)^5\right]. \tag{4.4}$$

It is thus seen that the low wavenumber transfer comes mostly from the 'beating' of two eddies with size of the order of the integral scale. This result (first communicated to us privately by J. R. Herring) may be considered as an alternative presentation of the non-constancy of the Loitsiansky integral (Proudman & Reid 1954). Indeed the Loitsiansky integral is just, within a numerical factor, the coefficient of k^4 in the Taylor expansion of the spectrum at $k = 0$ when this spectrum involves no lower powers of k.

A consequence is that, when the initial spectrum follows at low k a k^a law with $1 < a < 4$, the spectrum in this range remains essentially unchanged and we obtain by well-known arguments (presented, e.g., in TKM) an energy decay law:

$$E(t) \propto t^{-b}, \quad b = 2(a+1)/(a+3). \tag{4.5}$$

However, for $a = 4$, the coefficient of k^4 will change with time and the exponent of the energy decay cannot be easily predicted. (In particular (4.5) which gives $b = \frac{10}{7}$ for $a = 4$ is incorrect.) A similar phenomenon happens for Burgers' turbulence when $a = 2$. It is then possible to work out exactly, *without closure*, the law of decay of the energy for large t. For Gaussian initial conditions, Kida (1979) has thus shown that

$$E(t) \propto t^{-1} (\ln t)^{-\frac{1}{2}}$$

and not $t^{-\frac{6}{5}}$ as predicted by (3.2). For Navier–Stokes, the case $a = 4$ has been analysed in detail in LS for the EDQNM equation. The arguments remain essentially unchanged for the QNM equation. In particular it has been found that in the limit of infinite Reynolds numbers the spectrum tends for large times to a self-similar shape. At low wavenumbers the spectrum behaves like

$$E(k,t) \propto t^{\gamma} k^4, \tag{4.6}$$

where γ is a positive exponent solution of a nonlinear-eigenvalue problem which appears to be feasible only by numerical methods (see below). The asymptotic law of energy decay is then

$$E(t) \propto t^{-\frac{1}{7}(10-2\gamma)}. \tag{4.7}$$

These considerations have been tested numerically by integrating the QNM equation for an initial Reynolds number of 16 000.†

The numerical method is described in LS; it is a modification of a standard method used by Leith (1971), Orszag (1976) and André & Lesieur (1977). Their standard method, contrary to the method used by TKM, does not correctly represent the contribution of very elongated triads which are the dominant ones for small k. A modification introduced by LS allows for explicit representation of 'non-local' interaction obtained by analytically expanding the transfer integral in regions such that

$$k \ll p \sim q \quad \text{or} \quad p \ll k \sim q \quad \text{or} \quad q \ll k \sim p. \tag{4.8}$$

The results are displayed on figure 3. Figure 3 (a) corresponds to $a = 2$. It is seen that the spectrum remains essentially unchanged at low k. This is in agreement with TKM's result (their case I). Figure 3 (b) corresponds to $a = 4$ (their case II). Our numerical integrations which have been carried out for times much larger than in TKM indicate a substantial increase in the factor multiplying k^4. This, we stress again, comes from the back transfer of energy resulting from the beating of two energy-carrying eddies. We find that the correction $\frac{2}{7}\gamma$ changes the exponent b from $\frac{10}{7}$ to $\frac{10}{7} - 0{\cdot}04 \approx 1{\cdot}39$ in agreement with TKM's result (their equation (7.4)). Note that for EDQNM the correction is 0·05 (LS). For $a > 4$ the large time behaviour essentially reproduces the $a = 4$ case (LS). In figure 4 we have shown for comparison QNM and EDQNM results for $a = 4$ at $t = 100$ and initial Reynolds numbers 16 000.

As shown by LS, when the Reynolds number is not very high, complete self-similar decay of the spectrum is not expected; the reason being that the wavenumber k_0 characteristic of the energy-containing eddies does not have the same temporal variation as the wavenumber, k_D, characteristic of the dissipation. There is however an exceptional case for $a = 1$, where self-similar decay with unchanged Reynolds number $\sim (k_D/k_0)^{\frac{4}{3}}$ takes place and the energy follows a k^{-1} law.

5. The dissipation range

The question of the dissipation range has been studied analytically by Kraichnan (1959) and Orszag (1966). Kraichnan studied the direct-interaction equation and Orszag the EDQNM equation. All these equations reduce to the QNM equation in the dissipation range (eddy damping becomes negligible in comparison with viscous damping). They found that, for $k \gg k_D$,

$$E(k) \propto k^3 \exp(-k/k_D). \tag{5.1}$$

Since Kraichnan's (1959) remarks are very brief and Orszag's (1966) derivation is not published let us recall how they obtained equation (5.1).

Consider a solution of the QNM equation at a time $t \gg t_*$. We then have a quasi-steady inertial range and dissipation range, i.e. the characteristic time for changes in the energy spectrum at such wavenumbers is much longer than either the local eddy

† For decay calculations it is necessary to integrate up to about 100 initial turnover times of the energy-carrying eddies to clearly observe the self-similar regime; since the computation time increases considerably with the Reynolds number we have used a somewhat lower one than in §2.

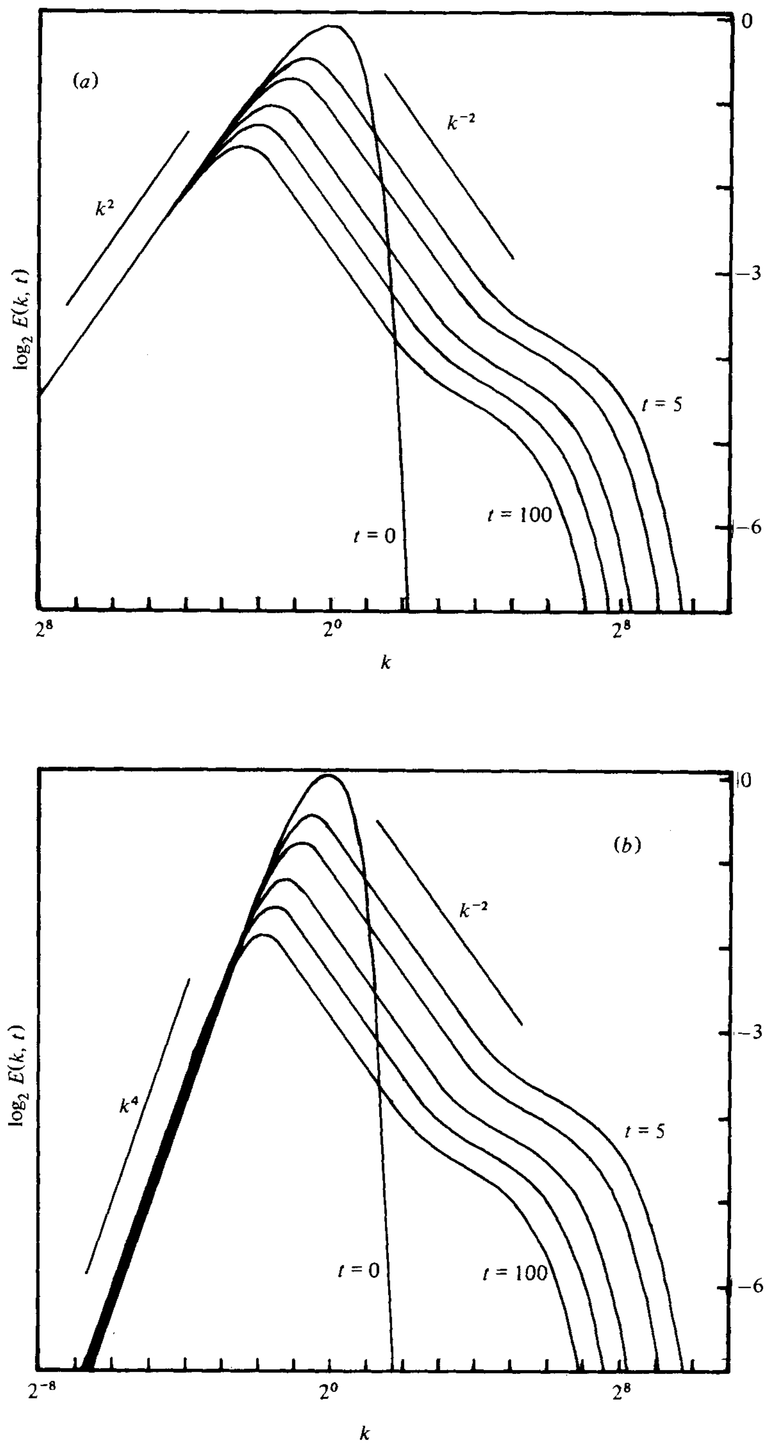

FIGURE 3. Self-similar decay of the energy spectrum at initial Reynolds number $R = 16000$. (*a*) Initial spectrum proportional to k^2 at low k. (*b*) Initial spectrum proportional to k^4 at low k.

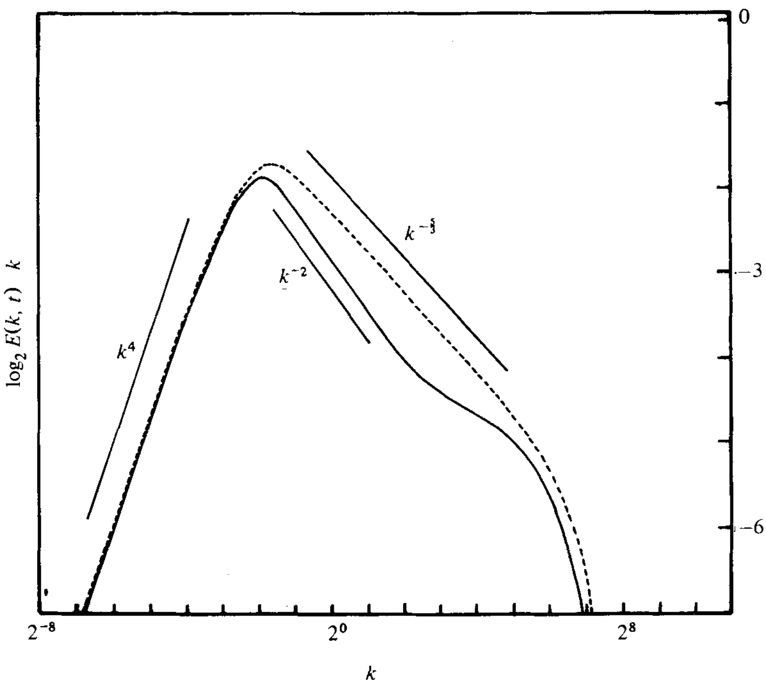

FIGURE 4. Comparison of energy spectra obtained at $t = 100$ with QNM (——) and EDQNM (– – –). Same initial energy spectrum as in figure 1. Initial $R = 16000$.

turnover time or the viscous decay time. Therefore, in the QNM equation the viscous term and the transfer term must essentially balance. Now it is easily shown that for $k \gg k_D$ the absorption term in the transfer integral (the $E(q,t)\,E(k,t)$ term) is negligible compared to the emission term (the $E(p,t)\,E(q,t)$ term). So the viscous term must balance the emission term. One then observes that for $k \gg k_D$ this balance cannot hold if the energy spectrum behaves like $\exp[-(k/k_D)^\alpha]$ with $\alpha > 1$. This follows from the convexity of the function $x \to x^\alpha$ which, together with $p+q \geqslant k$, implies $p^\alpha + q^\alpha \geqslant k^\alpha/2^\alpha$; as a consequence the emission term would be exponentially smaller than the viscous term. Similarly for $\alpha < 1$, it would be exponentially larger (after symmetrization in p and q the integrand becomes positive so that a lower bound of the integral is obtained by restricting p and q to be in the neighbourhood of $\frac{1}{2}k$). Now assume that the energy spectrum is of the form $\exp(-k/k_D)$ times some polynomial in k. Again from the inequality $p+q \geqslant k$, it is seen that the dominant contribution comes from *nearly flat* triads, i.e. such that $F = (p+q)/k - 1 \lesssim k_D/k \ll 1$. The result (5.1) is then obtained by performing an asymptotic expansion of the emission term near the $p+q = k$ edge of the Δ_k strip.

It must be stressed that TKM's as well as our numerical methods do correctly treat elongated triads but not nearly flat ones. For example in TKM's calculation local triads (p and q comparable to k) will have a minimum non-zero flatness F determined by the mesh size $\Delta\mu$ (see TKM's §5). It is then no longer possible to rule out an $\alpha > 1$ dissipation-range spectrum. It is not clear to us whether TKM's $\exp - \sigma k^{1.5}$ result is a numerical artifact or just corresponds to some intermediate dissipation range where the asymptotic formula (5.1) does not yet hold.

We are indebted to J.-P. Chollet for assistance in the numerical treatment of non-local interactions, to J. R. Herring for pointing out to us the importance of non-local interactions in the context of self-similar decay and to T. Bell for providing us with an independent proof of equation (5.1).

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Note added in proof October 1979

A detailed derivation of the exponential dissipation-range law (5.1) may be found in a recent paper by G. A. Kuzmin. Dissipation range of d-dimensional turbulence (1979). *Institut Teplophysiki Preprint* 39–79. Novosibirsk, U.S.S.R.